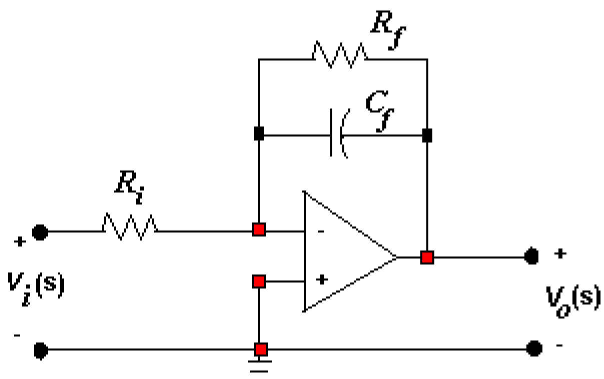
Note09: Active Filters ---Part 2

4. Higher-Order Active Filter

The first-order filters do not sharply divide the pass bands and the stop bands. ***One approach*** to obtain a sharper transition between the pass band and the stop band is to cascade identical filters two, three, or more. The sharpness of the cascaded filters can be clearly seen by a Bode plot.

Let's start from a first-order low pass filter we already discussed.

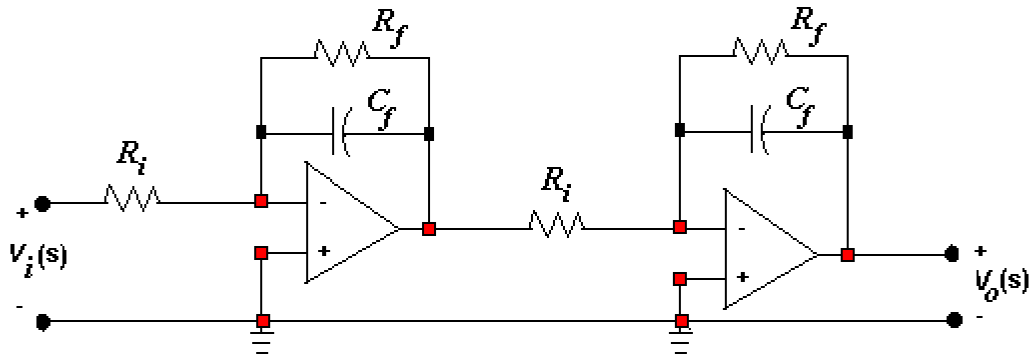


The transfer function of the above circuit is: $H(s) = -\frac{R_f}{R_i} \cdot \frac{1}{(1+R_f C_f s)}$

and the steady-state transfer function is: $H(jw) = -\frac{R_f}{R_i} \cdot \frac{1}{1+jR_f C_f w}$

The relative dB amplitude is: $A_{dB}(w) = 20\log(R_f/R_i) - 10\log[1+\left(\frac{w}{1/R_f C_f}\right)^2]$ -------(1)

Now let's connect one more identical filter circuit to the above circuit:



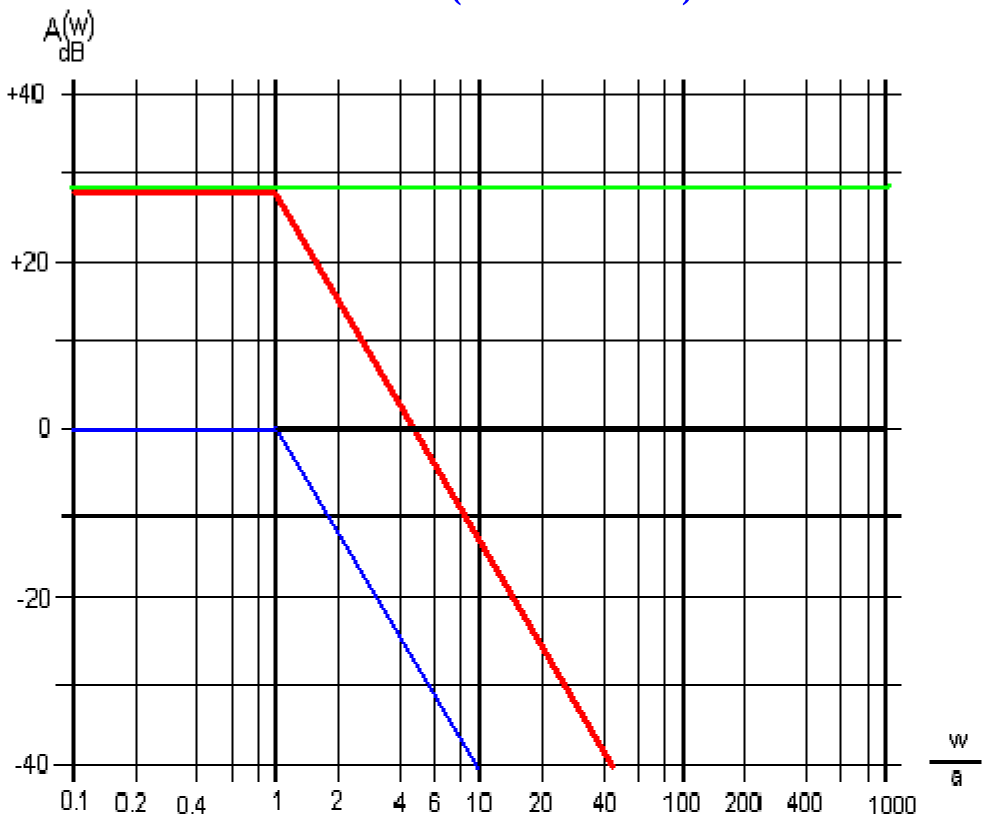
Then the transfer function becomes: $H(s) = (R_f/R_i)^2 \cdot \frac{1}{(1+R_f C_f s)} \cdot \frac{1}{(1+R_f C_f s)}$

The steady-state transfer function becomes: $H(jw) = (\frac{R_f}{R_i})^2 \cdot \frac{1}{(1+jR_fC_fw)^2}$

The relative dB amplitude becomes: $A_{dB}(w) = 40\log(R_f/R_i) - 20\log[1+\left(\frac{w}{1/R_fC_f}\right)^2]$

Let's draw two Bode plots with R_i=800, R_f=4000, and C_i=0.1uF for the ***first-order*** and the ***second-order*** (two cascaded first-order) filters.

Do you see the slope of 12dB/Octave (or 40 dB/decade) for the cascaded-filter, instead of the first-order filter's 6dB/Octave (or 20 dB/decade)?



Can we draw a general slope sharpness for *n*-element cascades?

20n dB/decade or 6n dB/Octave.

5. Higher-Order Active Filter (Butterworth Filters)

The other way to achieve higher order filter other than cascading identical filters.

1. British engineer S. Butterworth developed this type of filter.
2. This section may contain some materials that go above your current understanding of math or math related subject. Fortunately, your main job is to understand the result not the process. Bottom line: You may want to know only the item number 9, by the way.
3. A unity-gain Butterworth LPF has a transfer function whose magnitude is given by

$$|H(jw)| = \frac{1}{\sqrt{1+\left(\frac{w}{w_c}\right)^{2n}}},$$ n=order of the filter

4. A unity-gain Butterworth LPF has a transfer function whose magnitude is given by

$$|H(jw)| = \frac{s^{2n}}{\sqrt{1+\left(\frac{w}{w_c}\right)^{2n}}}$$

5.Let's consider the LPF case: Observations

(i) cutoff frequency w_c is all the same for all the values of n.

(ii) If n is large enough, the denominator is always close to 1 (when $w < w_c$)

(iii) The exponent of w/w_c is always even.

6. Then, the next big question is how to find a transfer function H(s) from the magnitude of the transfer function given as above. This derivation of H(s) involves a long process:

(i) Let's start from a prototype LPF with w_c=1 [rad/sec].

(ii) Then the LPF magnitude of transfer function becomes: $|H(jw)| = \frac{1}{\sqrt{1+w^{2n}}}$

(iii)By the way, H(jw) is the steady-state transfer function of a transfer function H(s), therefore, in frequency domain: H(jw)=H(s) and H(-jw)=H(-s).

Also from $s = jw$, $s^2 = -w^2$

(iv) Then, $|H(jw)|^2 = |H(jw)| \cdot |H(-jw)| = H(s) \cdot H(-s)$

(v) Also, $w^{2n} = [w^2]^n = [-s^2]^n = [(-1)s^2]^n = (-1)^n s^{2n}$

(vi)Therefore, $|H(jw)|^2 = \frac{1}{1+w^{2n}} = \frac{1}{1+(1)^n s^{2n}}$

(vii) Therefore, $H(s) \cdot H(-s) = \frac{1}{1+(1)^n s^{2n}}$ --------(1)

7. Finding H(s) from equation (1) is the discussion of another length process:

(i) Find the roots of the polynomial: $1+(-1)^n s^{2n} = 0$

(ii) Assign the left-half plane roots to H(s), and the right-plane roots to H(-s).

(iii) Combine terms in the denominator of H(s) to form first- and second-order factors.

8. Let's have an example of applying the process stated in 7 above.

(i)Q: Find the Butterworth transfer function for n=2 (2nd order Butterworth filter) (ii)The polynomial is then: $1+(-1)^2 s^4 = 0 \Rightarrow s^4 = -1 \Rightarrow s^2 = \pm j$

(iii) For $s^2 = j$, we have the following two solutions:

$$s_1 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \text{ and } s_2 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

(iv) For $s^2 = -j$, we have the following two solutions:

$$s_3 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \text{ and } s_4 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

(v) Therefore,

$$H(s)H(-s) = \frac{1}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} = \frac{1}{(s-s_1)(s-s_4)} \cdot \frac{1}{(s-s_2)(s-s_3)}$$

(vi) Finally, $H(s) = \dfrac{1}{(s+\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})(s+\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})} = \dfrac{1}{(s^2+\sqrt{2}s+1)}$

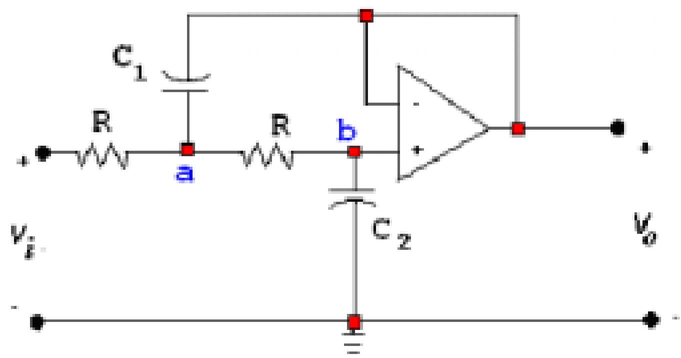
(9) For all *n*'s, we have the following Butterworth polynomials (H(s))

n	nth order of Butterworth Polynomial [H(s)]	
	LPF case	HPF case
1	$\frac{1}{(s+1)}$	$\frac{s}{(s+1)}$
2	$\frac{1}{(s^2+\sqrt{2}s+1)}$	$\frac{s^2}{(s^2+\sqrt{2}s+1)}$
3	$\frac{1}{(s+1)(s^2+s+1)}$	$\frac{s^3}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.847s+1)}$	$\frac{s^4}{(s^2+0.765s+1)(s^2+1.847s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$	$\frac{s^5}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$

(10) Now let's discuss about the implementation of a Butterworth filter. The first order Butterworth Filter building block is very simple: $\frac{1}{(s+1)}$ and $\frac{s}{(s+1)}$, and these two can be achieved easily from the very basic active filter circuits for LPF and HPF.

(11) Therefore, our major concern is how to implement a second order Butterworth filter building block has the following transfer functions for LPF and HPF, respectively: $\frac{1}{(s^2+b_1 s+1)}$ for LPF or $\frac{s^2}{(s^2+b_1 s+1)}$ for HPF.

(12) Let's discuss about $\frac{1}{(s^2+b_1 s+1)}$ implementation. See the following figure for a second order Butterworth LPF.



@node a: $\frac{V_a - V_i}{R} + \frac{V_a - V_o}{1/sC_1} + \frac{V_a - V_o}{R} = 0$ and @node b: $\frac{V_o}{1/sC_1} + \frac{V_o - V_a}{R} = 0$

Then, $H(s) = \frac{V_o}{V_i} = \frac{1/R^2C_1C_2}{s^2 + 2s/RC_1 + 1/R^2C_1C_2}$

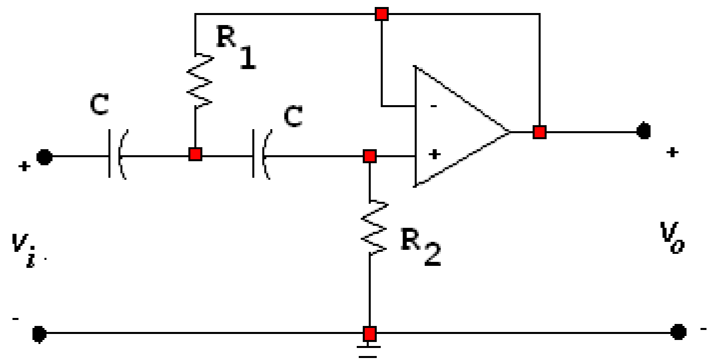
If we assume as prototype filter with R=1 and w_c=1, then the transfer function becomes:

$$H(s) = \frac{V_o}{V_i} = \frac{1/C_1C_2}{s^2 + 2s/C_1 + 1/C_1C_2}$$

Therefore, $b_1 = \frac{2}{C_1}$ and $1 = \frac{1}{C_1C_2}$

(13) Implementation of $\frac{s^2}{(s^2 + b_1 s + 1)}$. See the following circuit for the discussion.

Butterworth HPF



Similar node voltage equations lead to: $H(s) = \frac{V_o}{V_i} = \frac{s^2}{s^2 + 2s/R_2C + 1/CR_1R_2}$

Assuming a prototype filter with C=1, $H(s) = \frac{V_o}{V_i} = \frac{s^2}{s^2 + 2s/R_2 + 1/R_1R_2}$

Therefore, $b_1 = \frac{2}{R_2}$ and $1 = \frac{1}{R_1R_2}$

(14) EXAMPLE DESIGN for BUTTERWORTH FILTER: Design a 4th order Butterworth LPF with cutoff frequency of 500 Hz and passband gain of 10. Use as many 1K resistors as possible.

SOLUTION: