Class Note 02: Inverse Laplace Transformation

1. From the definition of Laplace transform, $L\{f(t)\} = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$, the inverse Laplace transform is given by,

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma 1 - j\infty}^{\sigma 1 + j\infty} F(s)e^{st}ds$$

where the integration is performed along a straight line (σ1+jw, for -∞ <w<∞) in the region of convergence, $\sigma 1 < \sigma_c$. This involves some knowledge about complex analysis beyond the scope of this course. For this reason, we use the properties of the Laplace transformation (see note 14) for inverse transformation. **In other words, we try to match function F(s) to an entry of note 14.**

2. Finding the inverse Laplace transform of F(s) involves two steps.
   (a) Decompose F(s) into simple terms using partial fraction expansion
   (b) Find the inverse of each term by matching the entries.
   *NOTE: Software packages such as Matlab, Mathcad, and Maple are capable of finding partial fraction expansions quite easily.

3. Simple Example

Find the inverse Laplace transform of $F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$

Solution: $f(t) = L^{-1}\{F(s)\} = L^{-1}\{\frac{3}{s}\} - L^{-1}\{\frac{5}{s+1}\} + L^{-1}\{\frac{6}{s^2+4}\} =$ ____________, t>0

4. Simple Real Root Example

Find f(t) given that $F(s) = \frac{s^2+12}{s(s+2)(s+3)}$

Solution:

Step 1: Partial fraction expansion: $F(s) = \frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$

Step 2: There are two approaches available.

(A)Residue Method: $A = sF(s)|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}|_{s=0} = 2$

$$B = (s+2)F(s)|_{s=-2} = \frac{s^2+12}{s(s+3)}|_{s=-2} = -8$$

$$C = (s+3)F(s)|_{s=-3} = \frac{s^2+12}{s(s+2)}|_{s=-3} = 7$$

(B)Algebraic Method: Multiplying both sides by $s(s+2)(s+3)$ gives

$$s^2+12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2) = (A+B+C)s^2 + (5A+3B+2C)s + 6A$$

Therefore, A+B+C=1, 5A+3B+2C=0, and 6A=12.

Finally, A=2, B=-8, and C=7

Step 3: From $F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$, $f(t) =$ ______________________________, t>0

5. Repeated Real Root Example

Calculate v(t) given that $V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$

Solution:

Step 1: $V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$

Step 2:

(A) Residue Method: $A = sV(s)|_{s=0} = \frac{10s^2+4}{(s+1)(s+2)^2}|_{s=0} = 1$

$$B = (s+1)V(s)|_{s=-1} = \frac{10s^2+4}{s(s+2)^2}|_{s=-1} = -14$$

$$C = (s+2)^2V(s)|_{s=-2} = \frac{10s^2+4}{s(s+1)}|_{s=-2} = 22$$

NOTE HERE------> $D = \frac{d}{ds}[(s+2)^2V(s)]|_{s=-2} = \frac{d}{ds}[\frac{10s^2+4}{s^2+s}]|_{s=-2} = 13$

(B) Algebraic Method: Multiplying both sides by $s(s+1)(s+2)^2$ gives

$$10s^2+4 = (A+B+D)s^3 + (5A+4B+C+3D)s^2 + (8A+4B+C+2D)s + 4A$$

Solving 4=4A, 0=8A+4B+C+2D, 10=5A+4B+C+3D, and 0=A+B+D gives A=1, B=-1, C=22, and D=13.

Step 3: From $V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{1}{s} - \frac{14}{s+1} + \frac{22}{(s+2)^2} + \frac{13}{s+2}$,

$f(t) =$ ________________________________, t>0

6. Complex Root Example

Calculate $i(t)$ given that $I(s)=\dfrac{20}{(s+3)(s^2+8s+25)}$.

Observation: I(s) has a pair of complex roots at $s^2+8s+25=0$ or $s=-4\pm j3$

(Note that: $s=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ for the equation of $as^2+bs+c=0$)

Solution:

Step 1: $I(s)=\dfrac{20}{(s+3)(s^2+8s+25)}=\dfrac{A}{s+3}+\dfrac{Bs+C}{s^2+8s+25}$

Step 2:

(a) Residue + Algebraic Method: $A=(s+3)I(s)|_{s=-3}=\dfrac{20}{s^2+8s+25}|_{s=-3}=2$

Then, let's substitute two specific value of s for two simultaneous equations.

s=0: $I(0)=\dfrac{20}{(3)(25)}=\dfrac{2}{3}+\dfrac{C}{25}$, therefore C= - 10

s=1: $I(1)=\dfrac{20}{(4)(1+8+25)}=\dfrac{2}{1+3}+\dfrac{B-10}{1+8+25}$, therefore B= - 2

(B) Algebraic Method: Multiplying both sides by $(s+3)(s^2+8s+25)$ gives

$$20=(A+B)s^2+(8A+3B+C)s+25A+3C$$

Finally, A=2, B=-2, and C=-10

Step 3: From $I(s)=\dfrac{20}{(s+3)(s^2+8s+25)}=\dfrac{2}{s+3}+\dfrac{-2s-10}{s^2+8s+25}$, we change the equation so that it matches with the entries of the properties of transformation.

$$I(s)=\frac{2}{s+3}+\frac{-2s-10}{s^2+8s+25}=\frac{2}{s+3}+\frac{-2(s+4)-2}{(s+4)^2+9}=\frac{2}{s+3}-\frac{-2(s+4)}{(s+4)^2+9}+\frac{2}{3}\frac{3}{(s+4)^2+3^2}$$

Therefore, $i(t)=$ ______________________________.

Applying the trigonometry

formulae of $A\cos x+B\sin x=\sqrt{A^2+B^2}\cos[x-\tan^{-1}\left(\frac{B}{A}\right)]$ gives,

$i(t)=$ ______________________________________